CSE 252B: Computer Vision II

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LECTURE 6
Stratification in 2D: From Projective to Affine to Euclidean

6.1. Introduction

The topic of this lecture is *stratified* reconstruction in the 2D case. Each *stratum* represents a different level of reconstruction we may wish to obtain, namely projective, affine and Euclidean. Although we will eventually do 3D reconstruction, the 2D case is easier to understand and gives us a preview of the 3D case. A general 2D homography can be decomposed into three components:

$$H = H_p H_a H_e = \underbrace{\begin{bmatrix} I & \mathbf{0} \\ \boldsymbol{v}^\top & 1 \end{bmatrix}}_{\text{projective}} \underbrace{\begin{bmatrix} K & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{affine}} \underbrace{\begin{bmatrix} R & \boldsymbol{T} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{Euclidean}} \tag{6.1}$$

Examples of affine and projective transformations on a plane are shown in Figure 1. Now we analyze H_e, H_a and H_p in more detail.

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(1) $H_e = \begin{bmatrix} R & \boldsymbol{T} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}$ is a 2D rigid transformation, where $R \in SO(2)$, $\boldsymbol{T} \in \mathbb{R}^2$.

(2) $H_a = \begin{bmatrix} K & \boldsymbol{0} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}$, where $K \in SL(2)/SO(2)$ is an affine transformation. $SL(2)$ is the special ($\det(K) = +1$) linear group of 2×2 matrices, and $SL(2)/SO(2)$ refers to $SL(2)$ specified up to a rotation. The ambiguity in K up to a rotation arises from the "camera frame/world frame" ambiguity. Therefore without loss of generality, we can take K to be upper-triangular by QR decomposition. Suppose $K = QR$, where R is an upper-triangular matrix[1], and $Q \in SO(2)$, then we can just set $K = R$.

(3) $H_p = \begin{bmatrix} I & \boldsymbol{0} \\ \boldsymbol{v}^\top & 1 \end{bmatrix}$ is a projective transformation known as an "elation." The presence of $\boldsymbol{v} \in \mathbb{R}^2$ makes H_p "interesting" in that it affects $\boldsymbol{\ell}_\infty = (0,0,1)^\top$, the line at infinity. Only H_p can map $\boldsymbol{\ell}_\infty$ to a finite line in the image plane, or vice versa.

6.2. Affine upgrade

Recall how points and lines are transformed under homography:

$$\boldsymbol{x}_2 = H\boldsymbol{x}_1, \qquad \boldsymbol{\ell}_2 = H^{-\top}\boldsymbol{\ell}_1$$

When H_a is a general affine transformation, we have

$$H_a = \begin{bmatrix} A & \boldsymbol{T} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}, \qquad H_a^{-\top} = \begin{bmatrix} A^{-\top} & \boldsymbol{0} \\ -\boldsymbol{T}^\top A^{-\top} & 1 \end{bmatrix}$$

with $A \in GL(2)$ and $\boldsymbol{T} \in \mathbb{R}^2$. It is straightforward to show that an affine transformation leaves $\boldsymbol{\ell}_\infty$ unchanged:

$$H_a^{-\top}\boldsymbol{\ell}_\infty \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{\ell}_\infty$$

(The mapping is not fixed pointwise, however.)

A projective transformation H_p *can* map $\boldsymbol{\ell}_\infty$ to a finite line, and vice versa. In fact, suppose the image of $\boldsymbol{\ell}_\infty$ is some line $\boldsymbol{\ell} = (a,b,c)^\top$, then the

[1]Note: the R of the QR decomposition is different notation from the usual R that denotes a rotation.

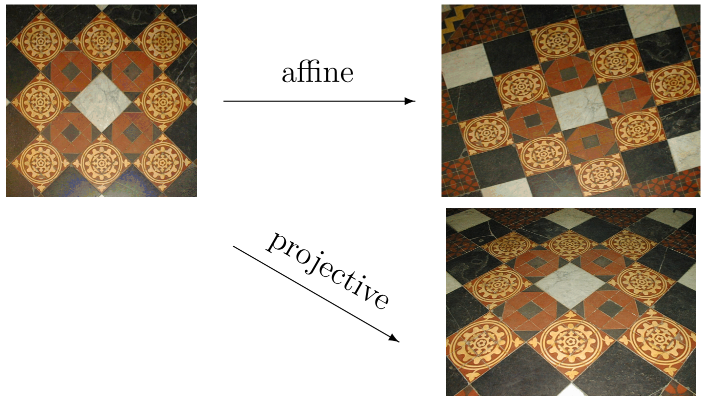

Figure 1. Affine vs. projective transformation. (From Hartley & Zisserman.)

following matrix H will send $\boldsymbol{\ell}$ back to infinity:

$$H = H_a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$$

where H_a is any affine transformation. We can show this by verifying that

$$H^{-\top}\boldsymbol{\ell} \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{\ell}_\infty$$

The affine upgrade process is illustrated in Figure 2. Parallelism is restored in the affine-upgraded image, but due to the presence of the unknown H_a, true angles between lines are lost.

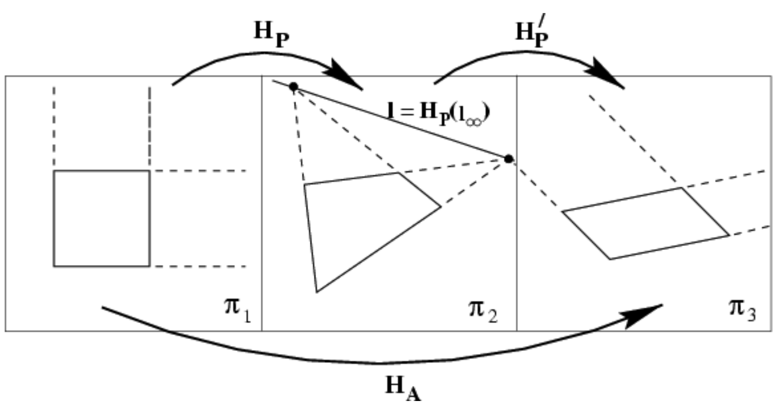

Figure 2. The affine upgrade

6.3. Euclidean upgrade

In the previous section, the key to the affine upgrade is the behavior of the line at infinity. The counterpart to this for the Euclidean upgrade is the behavior of the "circular points."

6.3.1. Circular points

Definition 6.2. The two circular points are defined as

$$\boldsymbol{I} = (1, i, 0)^\top \tag{6.3}$$

$$\boldsymbol{J} = (1, -i, 0)^\top \tag{6.4}$$

The circular points lie on $\boldsymbol{\ell}_\infty$, along with all other ideal points. All circles intersect $\boldsymbol{\ell}_\infty$ at points $\boldsymbol{I}$ and $\boldsymbol{J}$.

We can show that circular points are fixed points under homography H if and only if H is a similarity transformation. Let H_s be a similarity transformation, i.e. a rigid transformation with a constant scale factor $s \in \mathbb{R}$:

$$H_s = \begin{bmatrix} s\cos\theta & -s\sin\theta & T_x \\ s\sin\theta & s\cos\theta & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

Then it is straightforward to verify that

$$H_s\boldsymbol{I} = s\begin{bmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \\ 0 \end{bmatrix} = se^{-i\theta}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \sim \boldsymbol{I}$$

and similarly for $\boldsymbol{J}$.

The most intuitive way to upgrade from Affine to Euclidean is to see how a circle in the scene is transformed to an ellipse under the affine transformation, and find a transformation that maps it back to a circle. The upgrade is also possible when no actual circles are present in the scene, but for this we need to understand better how conics transform under a homography.

6.3.2. Transformation of conics under homography

Let's take a look at how conics transform under H. A conic is represented by

$$\boldsymbol{x}_1^\top C\boldsymbol{x}_1 = 0$$

Under homography H, the conic C is transformed into

$$\boldsymbol{x}_1^\top C\boldsymbol{x}_1 = \boldsymbol{x}_2^\top C'\boldsymbol{x}_2 = 0$$

where $C' = H^{-\top}CH^{-1}$.

6.3.3. Conics and dual conics

Recall the duality between points and lines,

$$\boldsymbol{x}_2 = H\boldsymbol{x}_1, \qquad \boldsymbol{\ell}_2 = H^{-\top}\boldsymbol{\ell}_1$$

Similarly, conics have "dual conics."

Definition 6.5. The dual of a conic C is the set of lines satisfying

$$\boldsymbol{\ell}^\top C^* \boldsymbol{\ell} = 0 \tag{6.6}$$

where C^* is the adjoint[2] of C. Figure 3 shows a conic and its dual.

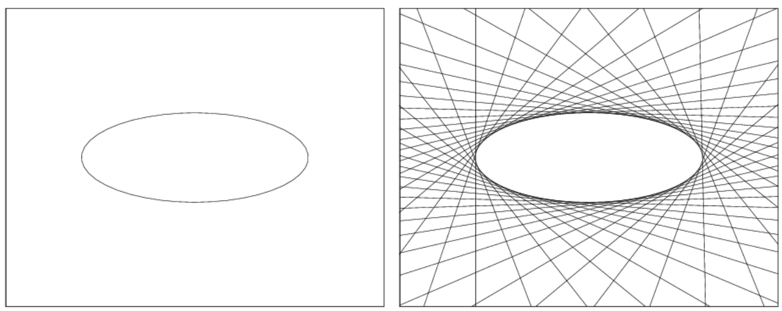

Figure 3. A conic and its dual conic

Dual conics tranform under homography H as

$$C^{*\prime} = HC^*H^\top$$

We can show that the line $\boldsymbol{\ell}$ tangent to C at a point $\boldsymbol{x}$ on the conic is given by

$$\boldsymbol{\ell} = C\boldsymbol{x}$$

Definition 6.7. The "conic dual to the circular points" is defined as

$$\begin{aligned} C^*_\infty &= \boldsymbol{I}\boldsymbol{J}^\top + \boldsymbol{J}\boldsymbol{I}^\top \\ &= \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -i & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \begin{bmatrix} 1 & i & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{6.8}$$

Notice that $\text{rank}(C^*_\infty) = 2$, which makes this is a degenerate conic.

[2]In this special case where C is symmetric and invertible, $C^* \sim C^{-1}$.

It is easy to verify that C^*_∞, the conic dual to the circular points, is fixed under homography H if and only if H is a similarity transformation. Let H_s be a similarity transformation. If we have $\boldsymbol{x}_2 = H_s\boldsymbol{x}_1$, then

$$C^{*\prime}_\infty = H_s C^*_\infty H_s^\top = C^*_\infty$$

6.3.4. Measuring Angles

Suppose we have identified C^*_∞, then the Euclidean angle θ between two lines $\boldsymbol{\ell}$ and $\boldsymbol{m}$ is given by

$$\cos\theta = \frac{\boldsymbol{\ell}^\top C^*_\infty \boldsymbol{m}}{\sqrt{\boldsymbol{\ell}^\top C^*_\infty \boldsymbol{\ell}}\sqrt{\boldsymbol{m}^\top C^*_\infty \boldsymbol{m}}} \tag{6.9}$$

It is easy to verify that this expression reduces to the usual expression for the angle between two lines in terms of the dot product between the normal vectors of each line in the Euclidean case. In particular, if we let $\boldsymbol{\ell} = (a, b, c)^\top$ and $\boldsymbol{m} = (d, e, f)^\top$ so that $\boldsymbol{\ell}^\top\boldsymbol{x} = ax+by+c = 0$ and $\boldsymbol{m}^\top\boldsymbol{x} = dx+ey+f = 0$, then this expression returns

$$\cos\theta = \frac{(a, b)^\top(d, e)}{\sqrt{a^2 + b^2}\sqrt{d^2 + e^2}}$$

This expression (6.9) for $\cos\theta$ is important because it is invariant under H. Given $\boldsymbol{x}_2 = H\boldsymbol{x}_1$, $\boldsymbol{\ell}_2 = H^{-\top}\boldsymbol{\ell}_1$, and $C^{*\prime}_\infty = HC^*_\infty H^\top$, then we can verify that

$$\boldsymbol{\ell}_2^\top C^{*\prime}_\infty \boldsymbol{m}_2 = \boldsymbol{\ell}_1 H^{-1} H C^*_\infty H^\top H^{-\top}\boldsymbol{m}_1 = \boldsymbol{\ell}_1^\top C^*_\infty \boldsymbol{m}_1$$

Corollary. Two lines $\boldsymbol{\ell}$ and $\boldsymbol{m}$ are orthogonal if

$$\boldsymbol{\ell}^\top C^*_\infty \boldsymbol{m} = 0$$

6.3.5. Solving for C^*_∞

Noting that $C^{*\prime}_\infty = HC^*_\infty H^\top$, and using $H = H_p H_a H_s$, we can obtain

$$\begin{aligned} C^{*\prime}_\infty &= (H_p H_a H_s) C^*_\infty (H_p H_a H_s)^\top \\ &= (H_p H_a) C^*_\infty (H_p H_a)^\top \\ &= \begin{bmatrix} KK^\top & KK^\top\boldsymbol{v} \\ \boldsymbol{v}^\top KK^\top & \boldsymbol{v}^\top KK^\top \boldsymbol{v} \end{bmatrix} \end{aligned} \tag{6.10}$$

Using knowledge of orthogonal line pairs in the scene, one can set up a homogeneous linear system to solve for $C^{*\prime}_\infty$ (as in Liebowitz and Zisserman [1998], or Hartley and Zisserman [2004] p. 56). An example of an upgrade from affine to Euclidean is given in Figure 4 using two pairs of orthogonal lines.

Furthermore, the SVD of ${C^*_\infty}'$ has the form

$${C^*_\infty}' = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^\top$$

Identifying the middle matrix as C^*_∞, we can set $H = U$ to upgrade directly from projective to Euclidean (assuming ${C^*_\infty}'$ is known).

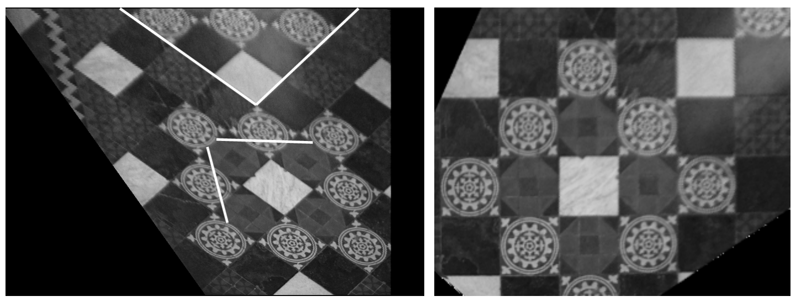

Figure 4. The Euclidean upgrade